

# Construction of Capacity-Achieving Lattice Codes: Polar Lattices

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**Abstract**—In this paper, we propose a new class of lattices constructed from polar codes, namely polar lattices, to achieve the capacity  $(1/2) \log(1 + \text{SNR})$  of the additive white Gaussian-noise (AWGN) channel. Our construction follows the multilevel approach of Forney *et al.*, where we construct a capacity-achieving polar code on each level. The component polar codes are shown to be naturally nested, thereby, fulfilling the requirement of the multilevel lattice construction. We prove that the polar lattices are AWGN-good. Furthermore, using the technique of source polarization, we propose discrete Gaussian shaping over the polar lattice to satisfy the power constraint. Both the construction and shaping are explicit, and the overall complexity of encoding and decoding is  $O(N \log N)$  for any fixed target error probability.

**Index Terms**—AWGN-good lattices, discrete Gaussian shaping, lattice codes, multilevel construction, polar codes.

## I. INTRODUCTION

FAST-DECODABLE, structured code achieving the capacity of the power-constrained additive white Gaussian-noise (AWGN) channel is a major goal of communication theory. Polar codes, proposed by Arikan in [1], can provably achieve the capacity of binary memoryless symmetric (BMS) channels. An attempt to construct polar codes for the AWGN channel was given in [2], based on nonbinary polar codes or on the technique for the multi-access channel. Although coded modulation using polar codes has been investigated in literature [3], [4], the AWGN channel capacity has not been achieved, to the best of our knowledge.

Lattice codes are counterparts of linear codes in the Euclidean space. The existence of lattice codes achieving the Gaussian channel capacity has been well established using the random coding argument [5], [6]. In the classical point-to-point channel, lattice codes offer a low-complexity solution compared to Gaussian random codes. More recently, thanks to their rich structures, lattice codes have emerged as a

novel framework of coding for multiuser communications, such as compute-and-forward [7], [8] and index coding [9]. In many problems of Gaussian multiuser networks, lattice codes demonstrate a clear advantage and outperform best known solutions based on random codes. This is because lattice codes enjoy the benefit of coordination despite the distributed nature of coding in a network. Readers are referred to [10, Ch. 12] for an extensive overview of the applications of lattice codes to Gaussian networks and their advantages over classical random coding approaches.

It is well known that the design of a lattice code consists of two essentially separate problems: AWGN coding and shaping. AWGN coding is addressed by the notion of AWGN-good lattices [5], [11]. Recently, several new lattice constructions with good performance have been introduced [12]–[14]. On the other hand, shaping takes care of the finite power constraint of the Gaussian channel. Capacity-achieving shaping techniques include Voronoi shaping [5] and lattice Gaussian shaping [6], [15], [16]. Despite these significant progresses, an explicit construction of lattice codes achieving the capacity of the Gaussian channel is still open (since this work was completed, we have become aware of the work [13] which shows low density Construction-A (LDA) lattices achieve capacity when the signal-to-noise ratio (SNR)  $> 1$  in magnitude).

In this paper, we settle this open problem by employing the powerful tool of polarization in lattice construction. The novel technical contribution of this work is two-fold:

- The construction of polar lattices and the proof of their AWGN-goodness. We follow the multilevel construction of Forney *et al.* [17], where for each level we build a polar code to achieve its capacity. We prove that the subchannels arising from some lattice partition chains are successively degraded, which guarantees that the component polar codes are naturally nested, as required by the multilevel construction. This compares favorably with existing multilevel constructions [12], where extra efforts are needed to nest the component codes.
- The Gaussian shaping technique for polar lattices in the power-constrained AWGN channel. This is based on source polarization and may be viewed as inverse source coding. Finally, our scheme is able to achieve the capacity  $\frac{1}{2} \log(1 + \text{SNR})$  with low-complexity multistage successive cancellation (SC) decoding for any given SNR. It is worth mentioning that our proposed shaping scheme is not only a practical implementation of lattice Gaussian shaping, but also an improvement in the sense that we successfully remove the restriction  $\text{SNR} > e$  in [6, Theorem 3].

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Overall, both source and channel polarization are employed in the construction, resulting in an integrated approach in the sense that error correction and shaping are performed by one single polar code on each level. Moreover, capacity is achieved with minimum mean-square error (MMSE) lattice decoding. The construction of polar lattices with Gaussian shaping is as explicit as that of polar codes themselves, and the complexity is quasilinear:  $O(N \log^2 N)$  for a sub-exponentially vanishing error probability and  $O(N \log N)$  for a fixed error probability, respectively.

All random variables (RVs) will be denoted by capital letters. For a set  $\mathcal{I}$ ,  $\mathcal{I}^c$  denotes its complement, and  $|\mathcal{I}|$  represents its cardinality. Following the notation of [1], we denote  $N$  independent uses of channel  $W$  by  $W^N$ . By channel combining and splitting, we get the combined channel  $W_N$  and the  $i$ -th subchannel  $W_N^{(i)}$ . Throughout this paper, we use the binary logarithm, denoted by  $\log$ , and information is measured in bits. We follow the standard asymptotic notation  $f(x) = O(g(x))$  if  $\limsup_{x \rightarrow \infty} |f(x)/g(x)| < \infty$ .

## II. BACKGROUND ON LATTICE CODING

A lattice is a discrete subgroup of  $\mathbb{R}^n$  which can be represented by

$$\Lambda = \{\lambda = Bx : x \in \mathbb{Z}^n\},$$

where the generator matrix  $B$  is assumed to be of full rank in this paper. The theta series of  $\Lambda$  is defined as

$$\Theta_\Lambda(\tau) = \sum_{\lambda \in \Lambda} e^{-\pi\tau\|\lambda\|^2}, \quad \tau > 0.$$

Readers are referred to the text [10] for basic definitions of lattices.

In this work, we are mostly concerned with the block error probability  $P_e(\Lambda, \sigma^2)$  of lattice decoding. It is the probability  $\mathbb{P}\{x \notin \mathcal{V}(\Lambda)\}$  that an  $n$ -dimensional independent and identically distributed (i.i.d.) Gaussian noise vector  $x$  with zero mean and variance  $\sigma^2$  per dimension falls outside the Voronoi region  $\mathcal{V}(\Lambda)$ . For an  $n$ -dimensional lattice  $\Lambda$ , the volume of a fundamental region is given by  $V(\Lambda) = |\det(B)|$ . Define the VNR by

$$\gamma_\Lambda(\sigma) \triangleq \frac{V(\Lambda)^{\frac{2}{n}}}{\sigma^2}.$$

A sequence of lattices  $\Lambda^{(N)}$  of increasing dimension  $N$  is AWGN-good if, for any fixed VNR greater than  $2\pi e$ ,

$$\lim_{N \rightarrow \infty} P_e(\Lambda^{(N)}, \sigma^2) = 0.$$

For  $\sigma > 0$  and  $c \in \mathbb{R}^n$ , the Gaussian distribution of mean  $c$  and variance  $\sigma^2$  is defined as

$$f_{\sigma,c}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{\|x-c\|^2}{2\sigma^2}},$$

for all  $x \in \mathbb{R}^n$ . For convenience, let  $f_\sigma(x) = f_{\sigma,0}(x)$ .

Given a lattice  $\Lambda$ , we define the  $\Lambda$ -periodic function as

$$f_{\sigma,\Lambda}(x) = \sum_{\lambda \in \Lambda} f_{\sigma,\lambda}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \sum_{\lambda \in \Lambda} e^{-\frac{\|x-\lambda\|^2}{2\sigma^2}},$$

for  $x \in \mathbb{R}^n$ . Note that  $f_{\sigma,\Lambda}(x)$  is a probability density if  $x$  is restricted to a fundamental region  $\mathcal{R}(\Lambda)$ . It is actually the probability density function (PDF) of the  $\Lambda$ -aliased Gaussian noise, i.e., the Gaussian noise after the mod- $\mathcal{R}(\Lambda)$  operation [17]. When  $\sigma$  is small, the effect of aliasing becomes insignificant and the  $\Lambda$ -aliased Gaussian density  $f_{\sigma,\Lambda}(x)$  approaches a Gaussian distribution. When  $\sigma$  is large,  $f_{\sigma,\Lambda}(x)$  approaches a uniform distribution. This phenomenon is characterized by the flatness factor, which is defined for  $\Lambda$  as [18]

$$\epsilon_\Lambda(\sigma) \triangleq \max_{x \in \mathcal{R}(\Lambda)} |V(\Lambda)f_{\sigma,\Lambda}(x) - 1|.$$

It can be interpreted as the maximum variation of  $f_{\sigma,\Lambda}(x)$  from the uniform distribution over  $\mathcal{R}(\Lambda)$ .

We define the *discrete Gaussian distribution* over  $\Lambda$  centered at  $c \in \mathbb{R}^n$  as the following discrete distribution taking values in  $\lambda \in \Lambda$ :

$$D_{\Lambda,\sigma,c}(\lambda) = \frac{f_{\sigma,c}(\lambda)}{f_{\sigma,c}(\Lambda)}, \quad \forall \lambda \in \Lambda,$$

where  $f_{\sigma,c}(\Lambda) \triangleq \sum_{\lambda \in \Lambda} f_{\sigma,c}(\lambda) = f_{\sigma,\Lambda}(c)$ . Again for convenience, we write  $D_{\Lambda,\sigma} = D_{\Lambda,\sigma,0}$ .

If the flatness factor is negligible, the discrete Gaussian distribution over a lattice preserves the capacity of the AWGN channel [6, Th. 2].

## III. CONSTRUCTION OF POLAR LATTICES

We now follow Forney *et al.*'s multilevel approach [17] to construct polar lattices.

### A. Forney *et al.*'s Construction

Given a sublattice  $\Lambda' \subset \Lambda$ , the quotient group  $\Lambda/\Lambda'$  induces a partition of  $\Lambda$  into equivalence classes modulo  $\Lambda'$ . We call  $\Lambda/\Lambda'$  a lattice partition [17]. The order of the partition is denoted by  $|\Lambda/\Lambda'|$ , which is equal to the number of cosets. If  $|\Lambda/\Lambda'| = 2$ , we call this a binary partition. Similarly, if  $\Lambda' \subseteq \Lambda_{r-1} \subseteq \dots \subseteq \Lambda_1 \subseteq \Lambda$  for  $r \geq 1$  is a chain of lattices with quotients  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'$ , then  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'$  is called an  $n$ -dimensional lattice partition chain. For each partition  $\Lambda_{\ell-1}/\Lambda_\ell$  ( $1 \leq \ell \leq r$  with convention  $\Lambda_0 = \Lambda$  and  $\Lambda_r = \Lambda'$ ), a code  $\mathcal{C}_\ell$  over  $\Lambda_{\ell-1}/\Lambda_\ell$  selects a sequence of representatives  $a_\ell$  for the cosets of  $\Lambda_\ell$ . Consequently, if each partition is a binary partition, the codes  $\mathcal{C}_\ell$  are binary codes.

Construction D requires a set of nested linear binary codes  $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots \subseteq \mathcal{C}_r$  [17]. Suppose  $\mathcal{C}_\ell$  has block length  $N$  and the number of information bits  $k_\ell$  for  $1 \leq \ell \leq r$ . Choose a basis  $\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_N$  such that  $\mathbf{g}_1, \dots, \mathbf{g}_{k_\ell}$  span  $\mathcal{C}_\ell$ . In this work, we focus on the one-dimensional partition chain  $\mathbb{Z}/2\mathbb{Z}/\dots/2^r\mathbb{Z}$  for the simplicity of presentation. Accordingly, the lattice  $L$  admits the form [17]

$$L = \left\{ \sum_{\ell=1}^r 2^{\ell-1} \sum_{i=1}^{k_\ell} u_\ell^i \mathbf{g}_i + 2^r \mathbb{Z}^N \mid u_\ell^i \in \{0, 1\} \right\} \quad (1)$$

where the addition is carried out in  $\mathbb{R}^N$ . The fundamental volume of a lattice obtained from this construction is given by the volume of the bottom lattice  $\Lambda'$  as

$$V(L) = 2^{-NRc} V(\Lambda')^N,$$

where  $R_C = \sum_{\ell=1}^r R_\ell = \frac{1}{N} \sum_{\ell=1}^r k_\ell$  denotes the sum rate of component codes.

A mod- $\Lambda$  Gaussian channel is a Gaussian channel with an input in  $\mathcal{V}(\Lambda)$  and with a mod- $\mathcal{V}(\Lambda)$  operator at the receiver front end [17]. The capacity of the mod- $\Lambda$  channel for noise variance  $\sigma^2$  is

$$C(\Lambda, \sigma^2) = \log V(\Lambda) - h(\Lambda, \sigma^2), \quad (2)$$

where  $h(\Lambda, \sigma^2)$  is the differential entropy of the  $\Lambda$ -aliased noise over  $\mathcal{V}(\Lambda)$ :

$$h(\Lambda, \sigma^2) = - \int_{\mathcal{V}(\Lambda)} f_{\sigma, \Lambda}(x) \log f_{\sigma, \Lambda}(x) dx.$$

Given lattice partition  $\Lambda/\Lambda'$ , the  $\Lambda/\Lambda'$  channel is a mod- $\Lambda'$  channel whose input is restricted to discrete lattice points in  $(\Lambda + a) \cap \mathcal{R}(\Lambda')$  for some translate  $a$ . The capacity of the  $\Lambda/\Lambda'$  channel is given by [17]

$$\begin{aligned} C(\Lambda/\Lambda', \sigma^2) &= C(\Lambda', \sigma^2) - C(\Lambda, \sigma^2) \\ &= h(\Lambda, \sigma^2) - h(\Lambda', \sigma^2) + \log(V(\Lambda')/V(\Lambda)). \end{aligned} \quad (3)$$

Further, if  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'$  is a lattice partition chain, then

$$C(\Lambda/\Lambda', \sigma^2) = C(\Lambda/\Lambda_1, \sigma^2) + \dots + C(\Lambda_{r-1}/\Lambda', \sigma^2). \quad (4)$$

The key idea of [17] is to use a good component code  $\mathcal{C}_\ell$  to achieve the capacity  $C(\Lambda_{\ell-1}/\Lambda_\ell, \sigma^2)$  for each level  $\ell = 1, 2, \dots, r$  in Construction D. For such a construction, the total decoding error probability with multistage decoding is bounded by

$$P_e(L, \sigma^2) \leq \sum_{\ell=1}^r P_e(\mathcal{C}_\ell, \sigma^2) + P_e((\Lambda')^N, \sigma^2). \quad (5)$$

To achieve a vanishing error probability, i.e., to make  $P_e(L, \sigma^2) \rightarrow 0$ , we need to choose the lattice  $\Lambda'$  such that  $P_e((\Lambda')^N, \sigma^2) \rightarrow 0$  and that all the codes  $\mathcal{C}_\ell$  for the  $\Lambda_{\ell-1}/\Lambda_\ell$  channels have error probabilities tending to zero.

Since  $V(L) = 2^{-NR_C} V(\Lambda')^N$ , the logarithmic VNR of  $L$  is

$$\begin{aligned} \log \left( \frac{\gamma_L(\sigma)}{2\pi e} \right) &= \log \frac{V(L)^{\frac{2}{nN}}}{2\pi e \sigma^2} \\ &= \log \frac{2^{-\frac{2}{n} R_C} V(\Lambda')^{\frac{2}{n}}}{2\pi e \sigma^2} \\ &= -\frac{2}{n} R_C + \frac{2}{n} \log V(\Lambda') - \log 2\pi e \sigma^2. \end{aligned} \quad (6)$$

Define

$$\begin{cases} \epsilon_1 = C(\Lambda, \sigma^2) \\ \epsilon_2 = h(\sigma^2) - h(\Lambda', \sigma^2) \\ \epsilon_3 = C(\Lambda/\Lambda', \sigma^2) - R_C = \sum_{\ell=1}^r C(\Lambda_{\ell-1}/\Lambda_\ell, \sigma^2) - R_\ell, \end{cases} \quad (7)$$

where  $h(\sigma^2) = \frac{n}{2} \log 2\pi e \sigma^2$  is the differential entropy of the Gaussian noise. We note that,  $\epsilon_1 \geq 0$  represents the capacity of the mod- $\Lambda$  channel,  $\epsilon_2 \geq 0$  (due to the data processing inequality) is the difference between the entropy of the Gaussian noise and that of the mod- $\Lambda'$  Gaussian noise, and  $\epsilon_3 \geq 0$  is the total capacity loss of component codes.

Then we have

$$\log \left( \frac{\gamma_L(\sigma)}{2\pi e} \right) = \frac{2}{n} (\epsilon_1 - \epsilon_2 + \epsilon_3).$$

Since  $\epsilon_2 \geq 0$ , we obtain the upper bound<sup>1</sup>

$$\log \left( \frac{\gamma_L(\sigma)}{2\pi e} \right) \leq \frac{2}{n} (\epsilon_1 + \epsilon_3). \quad (8)$$

Since  $\log \left( \frac{\gamma_L(\sigma)}{2\pi e} \right) = 0$  represents the Poltyrev capacity [11] [10, Th. 6.3.1], i.e., the capacity per unit volume of an unconstrained AWGN channel, the right hand side of (8) gives an upper bound on the gap to the Poltyrev capacity. The bound is equal to  $\frac{6.02}{n} (\epsilon_1 + \epsilon_3)$  decibels (dB), by conversion of the binary logarithm into the base-10 logarithm.

To approach the Poltyrev capacity, we would like to have  $P_e(L, \sigma^2) \rightarrow 0$  for any  $\log \left( \frac{\gamma_L(\sigma)}{2\pi e} \right) > 0$ . Thus, from (8), we need that both  $\epsilon_1$  and  $\epsilon_3$  are arbitrarily small. In the following lemma, we upper-bound  $\epsilon_1$  by the flatness factor  $\epsilon_\Lambda(\sigma)$  of the top lattice.

*Lemma 1:* The capacity  $C(\Lambda, \sigma^2)$  of the mod- $\Lambda$  channel is bounded by

$$C(\Lambda, \sigma^2) \leq \log(1 + \epsilon_\Lambda(\sigma)) \leq \log(e) \cdot \epsilon_\Lambda(\sigma). \quad (9)$$

*Proof:* By the definition of the flatness factor, we have

$$f_{\sigma, \Lambda}(x) \leq \frac{1 + \epsilon_\Lambda(\sigma)}{V(\Lambda)}.$$

Thus, the differential entropy of the mod- $\Lambda$  Gaussian noise is bounded by

$$\begin{aligned} h(\Lambda, \sigma^2) &= - \int_{\mathcal{V}(\Lambda)} f_{\sigma, \Lambda}(x) \log f_{\sigma, \Lambda}(x) dx \\ &\geq - \int_{\mathcal{V}(\Lambda)} f_{\sigma, \Lambda}(x) \log \frac{1 + \epsilon_\Lambda(\sigma)}{V(\Lambda)} dx \\ &= - \log \frac{1 + \epsilon_\Lambda(\sigma)}{V(\Lambda)} \\ &= \log V(\Lambda) - \log(1 + \epsilon_\Lambda(\sigma)). \end{aligned}$$

Therefore, from (2),  $C(\Lambda, \sigma^2)$  is bounded by  $\log(1 + \epsilon_\Lambda(\sigma))$ . The second inequality in (9) follows from the fact  $\log(1+x) = \log_2(e) \cdot \log_e(1+x) \leq \log(e) \cdot x$  for  $x > 0$ .  $\square$

Thus, we have the following design criteria:

- The top lattice  $\Lambda$  has a negligible flatness factor  $\epsilon_\Lambda(\sigma)$ .
- The bottom lattice  $\Lambda'$  has a small error probability  $P_e(\Lambda', \sigma^2)$ .
- Each component code  $\mathcal{C}_\ell$  is a capacity-approaching code for the  $\Lambda_{\ell-1}/\Lambda_\ell$  channel.

Asymptotically, the error probability of a polar code of codeword length  $N$  decreases approximately as  $O(2^{-\sqrt{N}})$  [19] and we may desire a similar form for the error probability of a polar lattice. In (5), we can let  $P_e((\Lambda')^N, \sigma^2)$  decrease exponentially by increasing the volume of the bottom lattice  $\Lambda'$  or equivalently by expanding the partition chain. More explicitly, the next lemma shows that the first two criteria

<sup>1</sup>It was shown in [17] that  $\epsilon_2 \approx \pi P_e(\Lambda', \sigma^2)$ , which is negligible compared to the other two terms.

can be satisfied by  $r$  growing with  $\log N$  (see Appendix A for a proof).

*Lemma 2:* Consider a partition chain  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'$ . There exists a sequence of numbers of levels  $r = O(\log N)$  such that  $\epsilon_\Lambda(\sigma) = O(e^{-N})$  and  $P_e(\Lambda', \sigma^2) = O(e^{-N})$ .

*Remark 1:* Lemma 2 is mostly of theoretical interest, e.g., for proving a partition chain with increasing levels is capacity achieving. In practical designs, if the target error probability is fixed, e.g.,  $P_e(L, \sigma^2) = 10^{-5}$ , a small number of levels will suffice. This is because one can choose a top lattice such that  $\epsilon_\Lambda(\sigma) \approx 10^{-2}$  and a bottom lattice such that  $P_e(\Lambda', \sigma^2) \approx 10^{-6}$ , for instance. In fact, it was shown in [17] that a two-level partition chain  $\mathbb{Z}/2\mathbb{Z}/4\mathbb{Z}$  is enough if  $n = 1$ , although slightly more levels are needed if  $n > 1$ . Readers are referred to [17] for more details and Section V for design examples.

### B. Polar Lattices

It is shown in [17] that the  $\Lambda_{\ell-1}/\Lambda_\ell$  channel is symmetric, and the optimum input distribution is uniform [17]. Since we use a binary partition  $\Lambda_{\ell-1}/\Lambda_\ell$ , the input  $X_\ell$  is binary for  $\ell \in \{1, 2, \dots, r\}$ . Associate  $X_\ell$  with representative  $a_\ell$  of the coset in the quotient group  $\Lambda_{\ell-1}/\Lambda_\ell$ . The fact that the  $\Lambda_{\ell-1}/\Lambda_\ell$  channel is a BMS channel allows a polar code to achieve its capacity.

Let  $Y$  denote the output of the AWGN channel. Given  $x_{1:\ell-1}$ , let  $\mathcal{A}_\ell(x_{1:\ell})$  denote the coset chosen by  $x_\ell$ , i.e.,  $\mathcal{A}_\ell(x_{1:\ell}) = a_1 + \dots + a_\ell + \Lambda_\ell$ . The conditional PDF of this channel with input  $x_\ell$  and output  $\bar{y}_\ell = y \bmod \Lambda_\ell$  is given by [17]

$$\begin{aligned} P_{\bar{Y}_\ell | X_\ell, X_{1:\ell-1}}(\bar{y}_\ell | x_\ell, x_{1:\ell-1}) &= f_{\sigma, \Lambda_\ell}(\bar{y}_\ell - a_1 - \dots - a_\ell) \\ &= \frac{1}{\sqrt{2\pi\sigma}} \sum_{a \in \mathcal{A}_\ell(x_{1:\ell})} \exp\left(-\frac{\|\bar{y}_\ell - a\|^2}{2\sigma^2}\right). \end{aligned} \quad (10)$$

*Definition 1:* (Channel degradation [20]): Consider two channels  $W_1 : \mathcal{X} \rightarrow \mathcal{Y}_1$  and  $W_2 : \mathcal{X} \rightarrow \mathcal{Y}_2$ . Channel  $W_1$  is said to be (stochastically) degraded with respect to  $W_2$  if there exists a channel  $Q : \mathcal{Y}_2 \rightarrow \mathcal{Y}_1$  such that

$$W_1(y_1|x) = \sum_{y_2 \in \mathcal{Y}_2} W_2(y_2|x)Q(y_1|y_2).$$

The proof of the following lemma is given in Appendix B.

*Lemma 3:* Consider a self-similar binary lattice partition chain  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'$ , in which we have  $\Lambda_\ell = T^\ell \Lambda$  for all  $\ell$ , with  $T = \alpha V$  for some scale factor  $\alpha > 1$  and orthogonal matrix  $V$ . Then, the  $\Lambda_{\ell-1}/\Lambda_\ell$  channel is degraded with respect to the  $\Lambda_\ell/\Lambda_{\ell+1}$  channel for  $1 \leq \ell \leq r-1$ .

Now, we recall some basics of polar codes. Let  $W(y|x)$  be a BMS channel with input alphabet  $\mathcal{X} = \{0, 1\}$  with a priori distribution Bernoulli(1/2) and output alphabet  $\mathcal{Y} \subseteq \mathbb{R}$ . Polar codes are block codes of length  $N = 2^m$  with input bits  $u^{1:N}$ . Let  $I(W)$  be the capacity of  $W$ . Given a rate  $R < I(W)$ , the information bits are indexed by a set of  $RN$  rows of the generator matrix  $G_N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes m}$ , where  $\otimes$  denotes the Kronecker product. This gives an  $N$ -dimensional channel

$W_N(y^{1:N} | u^{1:N})$ . The channel seen by each bit [1] is given by

$$W_N^{(i)}(y^{1:N}, u^{1:i-1} | u^i) = \sum_{u^{i+1:N} \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N(y^{1:N} | u^{1:N}).$$

Arikan proved that as  $N$  grows, each channel  $W_N^{(i)}$  approaches either an error-free channel or a completely noisy channel. The set of almost completely noisy (resp. almost error-free) subchannels is called the frozen set  $\mathcal{F}$  (resp. information set  $\mathcal{I}$ ). One sets  $u^i = 0$  for  $i \in \mathcal{F}$  and only sends information bits within  $\mathcal{I}$ .

Given a priori input distribution Bernoulli(1/2), the error probability of channel  $W$  with transition probability  $P_{Y|X}$  under maximum-likelihood decision is given by

$$P_e(W) = \frac{1}{2} \sum_y \min\{P_{Y|X}(y|0), P_{Y|X}(y|1)\}.$$

The Bhattacharyya parameter serves as an upper bound on  $P_e(W)$ .

*Definition 2 (Bhattacharyya Parameter for Symmetric Channel [1]):* Given a BMS channel  $W$  with transition probability  $P_{Y|X}$ , the Bhattacharyya parameter  $Z \in [0, 1]$  is defined as

$$Z(W) \triangleq \sum_y \sqrt{P_{Y|X}(y|0)P_{Y|X}(y|1)}.$$

The rule of SC decoding is defined as

$$\hat{u}^i = \begin{cases} 0 & i \in \mathcal{F} \text{ or } \frac{W_N^{(i)}(y^{1:N}, \hat{u}^{1:i-1}|0)}{W_N^{(i)}(y^{1:N}, \hat{u}^{1:i-1}|1)} \geq 1 \text{ when } i \in \mathcal{I}, \\ 1 & \text{otherwise.} \end{cases}$$

Let  $P_B$  denote the block error probability of a binary polar code under SC decoding. It has been proved in [1] that  $P_B$  can be upper-bounded by the sum of the decoding error probability of the genie-aided SC decoder for each information bit, i.e.,  $P_B \leq \sum_{i \in \mathcal{I}} Z(W_N^{(i)})$ . It is worth mentioning that there are some other decoding methods such as belief propagation decoding [21] and list decoding [22], which perform better than SC decoding. However, in this work, we focus on SC decoding because it is sufficient to show that polar lattices are able to achieve the capacity of AWGN channels.

It was shown in [19] and [23] that for any  $\beta < \frac{1}{2}$ ,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{N} \left| \{i : Z(W_N^{(i)}) < 2^{-N^\beta}\} \right| &= I(W) \\ \lim_{m \rightarrow \infty} \frac{1}{N} \left| \{i : I(W_N^{(i)}) > 1 - 2^{-N^\beta}\} \right| &= I(W). \end{aligned}$$

This means that the fraction of good channels approaches to  $I(W)$  as  $m \rightarrow \infty$ . Therefore, constructing polar codes is equivalent to choosing the good indices.

Let  $\mathcal{P}(N, k_\ell)$  denote the component polar code for the  $\Lambda_{\ell-1}/\Lambda_\ell$  partition channel ( $1 \leq \ell \leq r$ ), where  $k_\ell$  is the size of its information set and  $N$  is the block length. We stack them as in Construction D to build the polar lattice. The following lemma shows that these component codes are nested, which guarantees that the multilevel construction creates a lattice [17]. Two rules may be used to determine the component codes. One is the *capacity rule* [17], [24], where the channel indices are selected according to a threshold on the

mutual information. The other is the *equal-error-probability rule* [24], namely, the same error probability for each level, where we select the channel indices according to a threshold on the error probability or the Bhattacharyya parameter. The advantage of the equal-error-probability rule is that it gives an estimate of the error probability. For this reason, we use the equal-error-probability rule in this paper. It is well known that the polar codes constructed according to these two rules converge to each other as the block length goes to infinity [1].

*Lemma 4:* For the equal-error-probability rule based on either the error probability or the Bhattacharyya parameter, the component polar codes built in the multilevel construction are nested, i.e.,  $\mathcal{P}(N, k_1) \subseteq \mathcal{P}(N, k_2) \subseteq \dots \subseteq \mathcal{P}(N, k_r)$ .

*Proof:* Firstly, consider the equal-error-probability rule based on the Bhattacharyya parameter. By [23, Lemma 4.7], if a BMS channel  $V$  is a degraded version of  $W$ , then the subchannel  $V_N^{(i)}$  is also degraded with respect to  $W_N^{(i)}$  and  $Z(V_N^{(i)}) \geq Z(W_N^{(i)})$ . Let the threshold be  $2^{-N^\beta}$  for some  $\beta < 1/2$ . The codewords are generated by  $x^{1:N} = u^T G_{\mathcal{I}}$ , where  $G_{\mathcal{I}}$  is the submatrix of  $G$  whose rows are indexed by information set  $\mathcal{I}$ . The information sets for these two channels are respectively given by

$$\begin{cases} \mathcal{I}_W = \{i : Z(W_N^{(i)}) < 2^{-N^\beta}\}, \\ \mathcal{I}_V = \{i : Z(V_N^{(i)}) < 2^{-N^\beta}\}. \end{cases}$$

Due to the fact that  $Z(V_N^{(i)}) \geq Z(W_N^{(i)})$ , we have  $\mathcal{I}_V \subseteq \mathcal{I}_W$ . If we construct polar codes  $\mathcal{P}(N, |\mathcal{I}_W|)$  over  $W$  and  $\mathcal{P}(N, |\mathcal{I}_V|)$  over  $V$ ,  $G_{\mathcal{I}_V}$  is a submatrix of  $G_{\mathcal{I}_W}$ . Therefore  $\mathcal{P}(N, |\mathcal{I}_V|) \subseteq \mathcal{P}(N, |\mathcal{I}_W|)$ .

From Lemma 3, the channel of the  $\ell$ -th level is always degraded with respect to the channel of the  $(\ell + 1)$ -th level, and consequently,  $\mathcal{P}(N, k_\ell) \subseteq \mathcal{P}(N, k_{\ell+1})$ , for  $1 \leq \ell < r$ .

Then, consider the selection based on the error probability itself. The nesting relation still holds. This is because, by [25, Lemma 3],  $P_e(V_N^{(i)}) \geq P_e(W_N^{(i)})$  since  $V_N^{(i)}$  is degraded with respect to  $W_N^{(i)}$ .  $\square$

However, the complexity of exact code construction for a BMS channel with a continuous output alphabet appears to be exponential in the block length. A quantization method was proposed in [25] which transforms a BMS channel with a continuous output alphabet to one with a finite output alphabet. Also, the authors of [26] proposed an approximation method to construct polar codes efficiently over any BMS channel. We follow these methods to construct polar codes for the  $\Lambda_{\ell-1}/\Lambda_\ell$  channel. Details can be found in [27].

### C. AWGN Goodness

By combining the previous lemmas, we arrive at the main result of this section:

*Theorem 1:* Construct polar lattice  $L$  with the  $n$ -dimensional binary lattice partition chain  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'$  and  $r$  nested polar codes of block length  $N$ , where  $r = O(\log N)$  such that  $\epsilon_\Lambda(\sigma) = O(e^{-N})$  and  $P_e(\Lambda', \sigma^2) = O(e^{-N})$ . For any  $0 < \beta < 1/2$ , the error probability of  $L$

under multistage decoding is bounded by

$$P_e(L, \sigma^2) \leq rN2^{-N^\beta} + N \left( 1 - \int_{\mathcal{V}(\Lambda')} f_{\sigma^2}(x) dx \right), \quad (11)$$

with the logarithmic VNR bounded by (8). Then,  $L$  is AWGN-good, i.e.,  $P_e(L, \sigma^2) \rightarrow 0$  as  $N \rightarrow \infty$  for arbitrary VNR greater than  $2\pi e$ .

*Proof:* The fact that the component polar codes are nested is due to Lemma 4, while the condition  $r = O(\log N)$  is due to Lemma 2. The error probability bound (11) comes from (5). For a threshold  $2^{-N^\beta}$  of the Bhattacharyya parameter, the block error probability of a polar code with SC decoding is upper-bounded by  $N2^{-N^\beta}$ , which gives the first term on the right-hand side of (11). The second term of (11) is due to the union bound. Since both terms of (11) vanish as block length  $N$  grows,  $P_e(L, \sigma^2) \rightarrow 0$ .

Then we analyze the VNR. By Lemma 1, we have  $\epsilon_1 = C(\Lambda, \sigma^2) \leq \log(e) \cdot \epsilon_\Lambda(\sigma) = O(e^{-N})$ . Also, the capacity loss  $\epsilon_3$  can be arbitrarily small as  $N \rightarrow \infty$ . Plugging these into (8), we can make  $\log\left(\frac{\gamma_L(\sigma)}{2\pi e}\right)$  arbitrarily close to 0 as  $N \rightarrow \infty$ .  $\square$

*Remark 2:* In practice, if the target error probability is fixed (e.g.,  $10^{-5}$ ),  $r$  can be a small integer, namely,  $r$  does not have to scale as  $\log N$ . Thus, the essential condition is  $N \rightarrow \infty$ . Particularly, our example in Section IV shows that  $r = 2$  is sufficient for a target error probability around  $10^{-5}$  when  $\sigma = 0.3380$ .

## IV. POLARIZATION-BASED GAUSSIAN SHAPING

To achieve the capacity of the power-constrained Gaussian channel, we can apply Gaussian shaping over the polar lattice  $L$ . However, it appears difficult to do so directly. In this section, we will apply Gaussian shaping to the top lattice  $\Lambda$  instead, which is more friendly for implementation. This is motivated by [6, Th. 2], which implies that one may construct a capacity-achieving lattice code from a good constellation. More precisely, one may choose a low-dimensional top lattice such as  $\mathbb{Z}$  and  $\mathbb{Z}^2$  whose mutual information has a negligible gap to the channel capacity as bounded in [6, Theorem 2], and then construct a multilevel code to achieve the capacity. We will show that this strategy is equivalent to implementing Gaussian shaping over the AWGN-good polar lattice.

### A. Asymmetric Channels in Multilevel Lattice Coding

By [6, Theorem 2], we choose a constellation  $D_{\Lambda, \sigma_s}$  such that the flatness factor  $\epsilon_\Lambda(\tilde{\sigma})$  is negligible, where  $\tilde{\sigma} = \frac{\sigma_s \sigma}{\sqrt{\sigma_s^2 + \sigma^2}}$ . Let the binary partition chain  $\Lambda/\Lambda_1/\dots/\Lambda_{r-1}/\Lambda'/\dots$  be labelled by bits  $X_1, \dots, X_r, \dots$ . Then,  $D_{\Lambda, \sigma_s}$  induces a distribution  $P_{X_{1:r}}$  whose limit corresponds to  $D_{\Lambda, \sigma_s}$  as  $r \rightarrow \infty$ . An example for  $D_{\mathbb{Z}, \sigma_s}$  for  $\sigma_s = 3$  is shown in Figure 1. In this case, a shaping constellation with  $M = 32$  ( $r = 5$ ) points are actually sufficient, since the total probability of these points is rather close to 1.

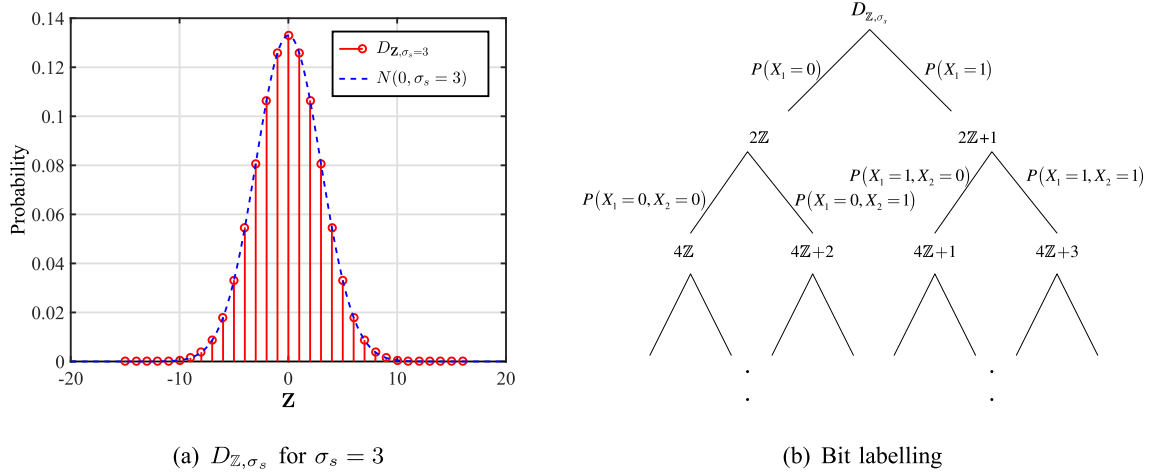


Fig. 1. Lattice Gaussian distribution  $D_{Z, \sigma_s}$  and the associated labelling. A probability  $P(X_1, X_2, \dots, X_i)$  in (b) is given by that of the coset indexed by bits  $X_1, X_2, \dots, X_i$ ; for example,  $P(X_1 = 1, X_2 = 0) = \sum_{\lambda \in 4Z+1} \Pr(\lambda)$ , where  $\Pr(\cdot)$  denotes the probability mass function of  $D_{Z, \sigma_s}$ .

By the chain rule of mutual information

$$I(Y; X_{1:r}) = \sum_{\ell=1}^r I(Y; X_\ell | X_{1:\ell-1}), \quad (12)$$

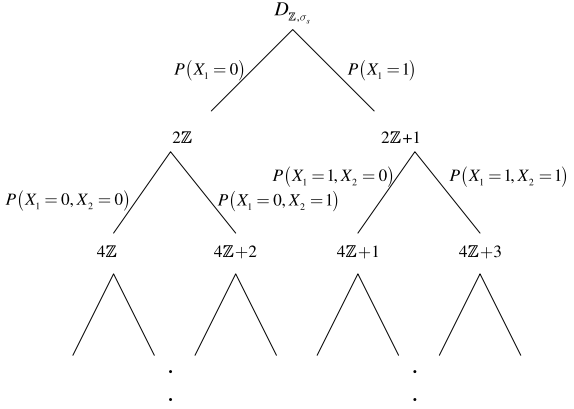
we obtain  $r$  binary-input channels  $W_\ell$  for  $1 \leq \ell \leq r$ . Given  $x_{1:\ell-1}$ , denote again by  $\mathcal{A}_\ell(x_{1:\ell})$  the coset of  $\Lambda_\ell$  indexed by  $x_{1:\ell-1}$  and  $x_\ell$ . According to [24], the channel transition PDF of the  $\ell$ -th channel  $W_\ell$  is given by

$$\begin{aligned} & P_{Y|X_\ell, X_{1:\ell-1}}(y|x_\ell, x_{1:\ell-1}) \\ &= \frac{1}{P\{\mathcal{A}_\ell(x_{1:\ell})\}} \sum_{a \in \mathcal{A}_\ell(x_{1:\ell})} P(a) P_{Y|A}(y|a) \\ &= \frac{1}{f_{\sigma_s}(\mathcal{A}_\ell(x_{1:\ell}))} \sum_{a \in \mathcal{A}_\ell(x_{1:\ell})} \frac{1}{2\pi\sigma_s} \exp\left(-\frac{\|y-a\|^2}{2\sigma^2} - \frac{\|a\|^2}{2\sigma_s^2}\right) \\ &= \exp\left(-\frac{\|y\|^2}{2(\sigma_s^2 + \sigma^2)}\right) \frac{1}{f_{\sigma_s}(\mathcal{A}_\ell(x_{1:\ell}))} \frac{1}{2\pi\sigma_s} \sum_{a \in \mathcal{A}_\ell(x_{1:\ell})} \\ &\quad \times \exp\left(-\frac{\sigma_s^2 + \sigma^2}{2\sigma_s^2\sigma^2} \left\| \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} y - a \right\|^2\right) \\ &= \exp\left(-\frac{\|y\|^2}{2(\sigma_s^2 + \sigma^2)}\right) \frac{1}{f_{\sigma_s}(\mathcal{A}_\ell(x_{1:\ell}))} \frac{1}{2\pi\sigma_s} \sum_{a \in \mathcal{A}_\ell(x_{1:\ell})} \\ &\quad \times \exp\left(-\frac{\|\alpha y - a\|^2}{2\sigma^2}\right). \end{aligned} \quad (13)$$

where  $\alpha = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2}$  is the MMSE coefficient. In general,  $W_\ell$  is asymmetric with the input distribution  $P_{X_\ell|X_{1:\ell-1}}$  unless  $f_{\sigma_s}(\mathcal{A}_\ell(x_{1:\ell}))/f_{\sigma_s}(\mathcal{A}_{\ell-1}(x_{1:\ell-1})) \approx \frac{1}{2}$ , which means that  $\epsilon_{\Lambda_\ell}(\sigma_s)$  is negligible.

For a finite power, the number of levels does not need to be large. The following lemma shows in a quantitative manner how large  $r$  should be in order to achieve the channel capacity. The proof can be found in [27].

**Lemma 5:** There exists  $r = O(\log \log N)$  such that using the first  $r$  levels only incurs a capacity loss  $\sum_{\ell > r} I(Y; X_\ell | X_{1:\ell-1}) = O(\frac{1}{N})$ .



**Remark 3:** The condition  $r = O(\log \log N)$  is of theoretical interest, similarly to the condition  $r = O(\log N)$  in the AWGN-good setting (Lemma 2). In practice,  $r$  can be a small constant so that the difference between  $I(Y; X_{1:r})$  and capacity is negligible, as we will see from the example in the next section. Note that the relaxed condition on  $r$  is thankfully due to the power constraint. Unlike the AWGN-good setting, here we no longer have the term  $P_e(\Lambda'^N, \sigma^2)$  in the upper bound of the error probability, since the bottom lattice  $\Lambda'$  does not carry any message.

### B. Polar Codes for Asymmetric Channels

Since the component channels are asymmetric, we need polar codes for asymmetric channels to achieve their capacity. Fortunately, polar codes for the binary memoryless asymmetric (BMA) channels have been introduced in [28] and [29] recently.

**Definition 3 (Bhattacharyya Parameter for BMA Channel [28], [30]):** Let  $W$  be a BMA channel with input  $X \in \mathcal{X} = \{0, 1\}$  and output  $Y \in \mathcal{Y}$ , and let  $P_X$  and  $P_{Y|X}$  denote the input distribution and channel transition probability, respectively. The Bhattacharyya parameter  $Z$  for channel  $W$  is defined as

$$\begin{aligned} Z(X|Y) &= 2 \sum_y P_Y(y) \sqrt{P_{X|Y}(0|y) P_{X|Y}(1|y)} \\ &= 2 \sum_y \sqrt{P_{X,Y}(0,y) P_{X,Y}(1,y)}. \end{aligned}$$

Note that this definition reduces to that for the BMS channel when  $P_X$  is uniform.

The next lemma shows that adding an observable at the output of  $W$  will not increase  $Z$ . Its proof is omitted due to space limitation, and can be found in [27].

**Lemma 6 (Conditioning Reduces Bhattacharyya Parameter  $Z$ ):** Let  $(X, Y, Y') \sim P_{X,Y,Y'}$ ,  $X \in \mathcal{X} = \{0, 1\}$ ,  $Y \in \mathcal{Y}$ ,  $Y' \in \mathcal{Y}'$ , we have

$$Z(X|Y, Y') \leq Z(X|Y).$$

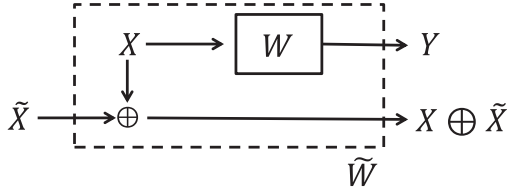


Fig. 2. The relationship between the asymmetric channel  $W$  and the symmetrized channel  $\tilde{W}$ .

Let  $X^{1:N}$  and  $Y^{1:N}$  be the input and output vector after  $N$  independent uses of  $W$ . For simplicity, denote the distribution of  $(X^i, Y^i)$  by  $P_{XY} = P_X P_{Y|X}$  for  $i \in [N]$ . The following property of the polarized random variables  $U^{1:N} = X^{1:N} G_N$  is well known.

*Theorem 2 (Polarization of Random Variables [28]):* For any  $\beta \in (0, 1/2)$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ i : Z(U^i | U^{1:i-1}) \geq 1 - 2^{-N^\beta} \right\} \right| &= H(X), \\ \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ i : Z(U^i | U^{1:i-1}) \leq 2^{-N^\beta} \right\} \right| &= 1 - H(X), \\ \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ i : Z(U^i | U^{1:i-1}, Y^{1:N}) \geq 1 - 2^{-N^\beta} \right\} \right| &= H(X|Y), \\ \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ i : Z(U^i | U^{1:i-1}, Y^{1:N}) \leq 2^{-N^\beta} \right\} \right| &= 1 - H(X|Y), \end{aligned} \quad (14)$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ i : Z(U^i | U^{1:i-1}, Y^{1:N}) \leq 2^{-N^\beta} \right. \right. \\ \left. \left. \text{and } Z(U^i | U^{1:i-1}) \geq 1 - 2^{-N^\beta} \right\} \right| &= I(X; Y), \\ \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ i : Z(U^i | U^{1:i-1}, Y^{1:N}) \geq 2^{-N^\beta} \right. \right. \\ \left. \left. \text{or } Z(U^i | U^{1:i-1}) \leq 1 - 2^{-N^\beta} \right\} \right| &= 1 - I(X; Y). \end{aligned} \quad (15)$$

The Bhattacharyya parameter for asymmetric models was originally defined for distributed source coding in [30]. By the duality between channel coding and source coding, it can be also used to construct capacity-achieving polar codes for BMA channels [28]. Actually,  $Z(U^i | U^{1:i-1})$  is the Bhattacharyya parameter for a single source  $X$  (without side information).

The Bhattacharyya parameter of a BMA channel can be related to that of a symmetric channel. To this aim, we use a symmetrization technique which creates a BMS channel  $\tilde{W}$  from the BMA channel  $W$  [23], [28].

*Lemma 7: (Symmetrization):* Let  $\tilde{W}$  be a binary-input channel with input  $\tilde{X} \in \mathcal{X} = \{0, 1\}$  and output  $\tilde{Y} \in \mathcal{Y} \times \mathcal{X}$ , built from the asymmetric channel  $W$  by treating  $\tilde{X}$  as the new input and  $\tilde{X} \oplus X$  as an additional output, as shown in Figure 2. Then  $\tilde{W}$  is a binary-input symmetric channel in the sense that  $P_{\tilde{Y}|\tilde{X}}(y, x \oplus \tilde{x}|\tilde{x}) = P_{Y,X}(y, x)$ . Therefore, the optimal input distribution of  $\tilde{W}$  is the uniform distribution.

The following theorem connects the Bhattacharyya Parameter of a BMA channel  $W$  and that of the symmetrized channel  $\tilde{W}$ . Denote by  $W_N$  and  $\tilde{W}_N$  the combining channels of  $N$  uses of  $W$  and  $\tilde{W}$ , respectively.

*Theorem 3: (Connection Between Bhattacharyya Parameters [28]):* Let  $\tilde{X}^{1:N}$  and  $\tilde{Y}^{1:N} = (Y^{1:N}, X^{1:N} \oplus \tilde{X}^{1:N})$  be the input and output vectors of  $\tilde{W}$ , respectively, and let  $U^{1:N} = X^{1:N} G_N$  and  $\tilde{U}^{1:N} = \tilde{X}^{1:N} G_N$ . The Bhattacharyya parameter of each subchannel of  $W_N$  is equal to that of each subchannel of  $\tilde{W}_N$ , i.e.,

$$\begin{aligned} Z(U^i | U^{1:i-1}, Y^{1:N}) &= \tilde{Z}(\tilde{W}_N^{(i)}) \\ &= Z(\tilde{U}^i | \tilde{U}^{1:i-1}, Y^{1:N}, X^{1:N} \oplus \tilde{X}^{1:N}). \end{aligned}$$

Now, we are in a position to construct polar codes for the BMA channel. Define the frozen set  $\tilde{\mathcal{F}}$  and information set  $\tilde{\mathcal{I}}$  of the symmetric polar codes as follows:

$$\begin{cases} \text{frozen set: } \tilde{\mathcal{F}} \\ \quad = \{i \in [N] : Z(U^i | U^{1:i-1}, Y^{1:N}) > 2^{-N^\beta}\} \\ \text{information set: } \tilde{\mathcal{I}} \\ \quad = \{i \in [N] : Z(U^i | U^{1:i-1}, Y^{1:N}) \leq 2^{-N^\beta}\}. \end{cases} \quad (16)$$

By Theorem 3, the Bhattacharyya parameters of the symmetrized channel  $\tilde{W}$  and the asymmetric channel  $W$  are the same. However, the channel capacity of  $\tilde{W}$  is  $I(\tilde{X}; X \oplus \tilde{X}) + I(\tilde{X}; Y | X \oplus \tilde{X}) = 1 - H(X) + I(X; Y)$ , which is  $1 - H(X)$  more than the capacity of  $W$ . To obtain the real capacity  $I(X; Y)$  of  $W$ , the input distribution of  $W$  needs to be adjusted to  $P_X$ . By polar lossless source coding, the indices with very small  $Z(U^i | U^{1:i-1})$  should be removed from the information set  $\tilde{\mathcal{I}}$  of the symmetrized channel, and the proportion of this part is  $1 - H(X)$  as  $N \rightarrow \infty$ . We name the remaining set as the information set  $\mathcal{I}$  of the asymmetric channel  $W$ . Further, there are some bits which are uniformly distributed and can be made independent from the information bits; we name this set as the frozen set  $\mathcal{F}$ . In order to generate the desired input distribution  $P_X$ , the remaining bits are determined by the bits in  $\mathcal{F} \cup \mathcal{I}$ ; we call it the shaping set  $\mathcal{S}$ . This process is depicted in Figure 3. We formally define the three sets as follows:

$$\begin{cases} \text{frozen set: } \mathcal{F} \\ \quad = \{i \in [N] : Z(U^i | U^{1:i-1}, Y^{1:N}) \geq 1 - 2^{-N^\beta}\} \\ \text{information set: } \mathcal{I} \\ \quad = \{i \in [N] : Z(U^i | U^{1:i-1}, Y^{1:N}) \leq 2^{-N^\beta} \text{ and } \\ \quad \quad Z(U^i | U^{1:i-1}) \geq 1 - 2^{-N^\beta}\} \\ \text{shaping set: } \mathcal{S} = (\mathcal{F} \cup \mathcal{I})^c. \end{cases} \quad (17)$$

To find these sets, one can use Theorem 3 to calculate  $Z(U^i | U^{1:i-1}, Y^{1:N})$  with the known technique for symmetric polar codes [25]. We note that  $Z(U^i | U^{1:i-1})$  can be computed in a similar way: one constructs a symmetric channel between  $\tilde{X}$  and  $X \oplus \tilde{X}$ , which is actually a binary-input symmetric channel with cross-over probability  $P_X(x = 1)$ . The above construction is equivalent to implementing shaping over the polar code for the symmetrized channel  $\tilde{W}$ .

Besides the construction, the decoding can also be converted to that of the symmetric polar code. If  $X^{1:N} \oplus \tilde{X}^{1:N} = 0$ , we have  $U^{1:N} = \tilde{U}^{1:N}$ , which means the decoding result of  $U^{1:N}$  equals to that of  $\tilde{U}^{1:N}$ . Thus, decoding of the polar code

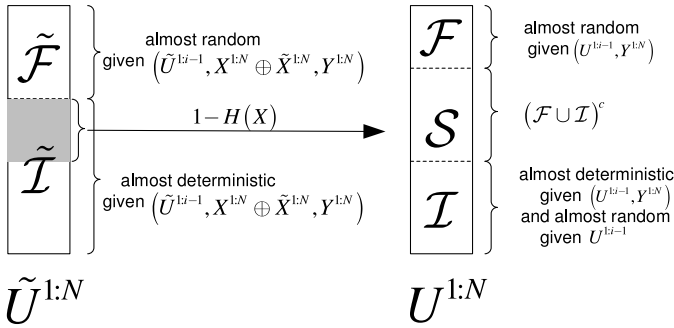


Fig. 3. Polarization for symmetric and asymmetric channels.

for  $W$  can be treated as decoding of the polar code for  $\tilde{W}$  given that  $X \oplus \tilde{X} = 0$ . Clearly, the SC decoding complexity for asymmetric channel is also  $O(N \log N)$ . We summarize this observation as the following lemma. The proof can be found in [28].

*Lemma 8: (Decoding for Asymmetric Channel [28]):* Let  $y^{1:N}$  be a realization of  $Y^{1:N}$  and  $\hat{u}^{1:i-1}$  be the previous  $i-1$  estimates of  $u^{1:N}$ . The likelihood ratio of  $u^i$  is given by

$$\frac{P_{U^i|U^{1:i-1}, Y^{1:N}}(0|\hat{u}^{1:i-1}, y^{1:N})}{P_{U^i|U^{1:i-1}, Y^{1:N}}(1|\hat{u}^{1:i-1}, y^{1:N})} = \frac{\tilde{W}_N^{(i)}((y^{1:N}, 0^{1:N}), \hat{u}^{1:i-1}|0)}{\tilde{W}_N^{(i)}((y^{1:N}, 0^{1:N}), \hat{u}^{1:i-1}|1)}, \quad (18)$$

where  $\tilde{W}_N^{(i)}$  denotes the transition probability of the  $i$ -th subchannel of  $\tilde{W}_N$ .

In [28], the bits in  $\mathcal{F} \cup \mathcal{S}$  are all chosen according to  $P_{U^i|U^{1:i-1}}(u^i|u^{1:i-1})$ , which can also be calculated using (18) (treating  $Y$  as an independent variable and remove it). However, in order to be compatible with polar lattices, we modify the scheme such that the bits in  $\mathcal{F}$  are uniformly distributed over  $\{0, 1\}$  while the bits in  $\mathcal{S}$  are still chosen according to  $P_{U^i|U^{1:i-1}}(u^i|u^{1:i-1})$ . The expectation of the decoding error probability still vanishes with  $N$ . The following theorem is an extension of the result in [28, Th. 3].

*Theorem 4:* Consider a polar code with the following encoding and decoding strategies for a BMA channel.

- Encoding: Before sending the codeword  $x^{1:N} = u^{1:N}G_N$ , the index set  $[N]$  are divided into three parts: the frozen set  $\mathcal{F}$ , the information set  $\mathcal{I}$  and the shaping set  $\mathcal{S}$  which are defined in (17). The encoder places uniformly distributed information bits in  $\mathcal{I}$ , and fills  $\mathcal{F}$  with a uniform random  $\{0, 1\}$  sequence which is shared between the encoder and the decoder. The bits in  $\mathcal{S}$  are generated by a mapping  $\phi_S \triangleq \{\phi_i\}_{i \in \mathcal{S}}$  in the family of randomized mappings  $\Phi_S$ , which yields the following distribution:

$$u^i = \phi_i(u^{1:i-1}) = \begin{cases} 0 & \text{with probability } P_{U^i|U^{1:i-1}}(0|u^{1:i-1}), \\ 1 & \text{with probability } P_{U^i|U^{1:i-1}}(1|u^{1:i-1}). \end{cases}$$

- Decoding: The decoder receives  $y^{1:N}$  and estimates  $\hat{u}^{1:N}$  of  $u^{1:N}$  according to the rule

$$\hat{u}^i = \begin{cases} u^i, & \text{if } i \in \mathcal{F} \\ \phi_i(\hat{u}^{1:i-1}), & \text{if } i \in \mathcal{S} \\ \underset{u}{\operatorname{argmax}} P_{U^i|U^{1:i-1}, Y^{1:N}} \\ (u|\hat{u}^{1:i-1}, y^{1:N}), & \text{if } i \in \mathcal{I}. \end{cases}$$

With the above encoding and decoding, the message rate can be arbitrarily close to  $I(X; Y)$  and the expectation of the decoding error probability over the randomized mappings satisfies  $E_{\Phi_S}[P_e^{SC}(\phi_S)] = O(2^{-N^{\beta'}})$  for  $\beta' < \beta < 0.5$ , where  $\beta$  is used to choose the frozen set, the information set, and the shaping set as in (17).

By an averaging argument, there exists a deterministic mapping  $\phi_S$  such that  $P_e(\phi_S) = O(2^{-N^{\beta'}})$ . However, it is difficult to actually find such a deterministic mapping. In practice, we may share a random mapping  $\phi_S$  between the encoder and decoder, i.e., let them have access to the same source of randomness (e.g., using the same seed for the pseudorandom number generators).

### C. Multilevel Polar Codes

Next, our task is to construct polar codes to achieve the mutual information  $I(Y; X_\ell | X_{1:\ell-1})$  for all levels. The construction of the preceding subsection is readily applicable to the construction for the first level  $W_1$ . To demonstrate the construction for other levels, we take the channel of the second level  $W_2$  as an example. This is also a BMA channel with input  $X_2 \in \mathcal{X} = \{0, 1\}$ , output  $Y \in \mathcal{Y}$  and side information  $X_1$ . Its channel transition probability is shown in (13). To construct a polar code for the second level, we propose the following two-step procedure.

Step 1:

- 1) Construct a polar code for the BMS channel with input vector  $\tilde{X}_2^{1:N} = [\tilde{X}_2^1, \tilde{X}_2^2, \dots, \tilde{X}_2^N]$  and output vector  $\tilde{Y}^{1:N} = (X_2^{1:N} \oplus \tilde{X}_2^{1:N}, Y^{1:N}, X_1^{1:N})$  where  $\tilde{X}_2^i \in \mathcal{X} = \{0, 1\}$  is uniformly distributed. At this step  $X_1$  is regarded as a part of the outputs. Then the distribution of  $X_2$  becomes the marginal distribution  $\sum_{x_1, x_{3:r}} P_{X_{1:r}}(x_{1:r})$ . Consider polarized random variables  $U_2^{1:N} = X_2^{1:N}G_N$  and  $\tilde{U}_2^{1:N} = \tilde{X}_2^{1:N}G_N$ . According to Theorem 2, the polarization gives us the three sets  $\mathcal{F}_2$ ,  $\mathcal{I}'_2$  and  $\mathcal{S}'_2$  as shown in Figure 4. Similarly, we can prove that  $\frac{|\mathcal{I}'_2|}{N} \rightarrow I(Y, X_1; X_2)$  and  $\frac{|\mathcal{F}_2 \cup \mathcal{S}'_2|}{N} \rightarrow 1 - I(Y, X_1; X_2)$  as  $N \rightarrow \infty$ . These three sets are defined as follows:

$$\begin{cases} \text{frozen set: } \mathcal{F}_2 = \{i \in [N] : \\ Z(U_2^i|U_2^{1:i-1}, Y^{1:N}, X_1^{1:N}) \geq 1 - 2^{-N^\beta}\} \\ \text{information set: } \mathcal{I}'_2 \\ = \{i \in [N] : Z(U_2^i|U_2^{1:i-1}, Y^{1:N}, X_1^{1:N}) \leq 2^{-N^\beta} \\ \text{and } Z(U_2^i|U_2^{1:i-1}) \geq 1 - 2^{-N^\beta}\} \\ \text{shaping set: } \mathcal{S}'_2 = (\mathcal{F}_2 \cup \mathcal{I}'_2)^c. \end{cases} \quad (19)$$



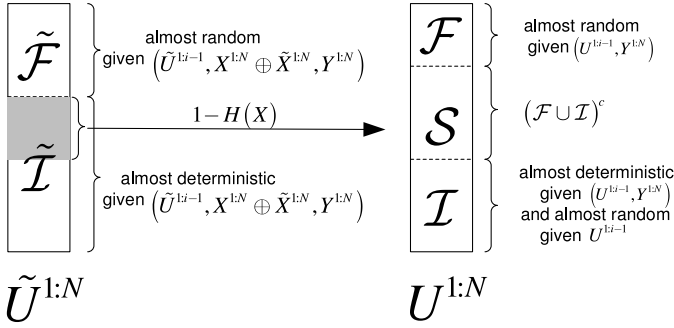


Fig. 4. The first step of polarization in the construction for the second level.

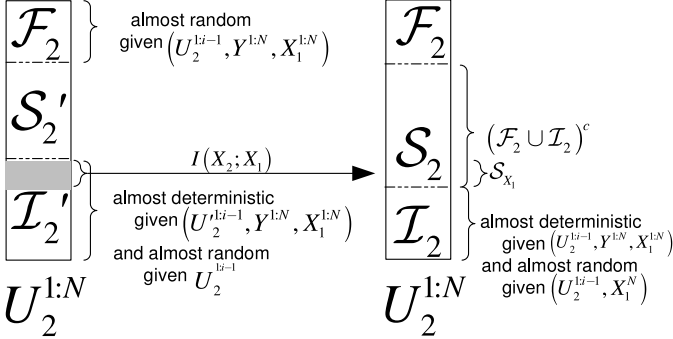


Fig. 5. The second step of polarization in the construction for the second level.

2) Treat  $X_1^{1:N}$  as the side information for the encoder. Given  $X_1^{1:N}$ , the choices of  $X_2^{1:N}$  are further restricted since  $X_1$  and  $X_2$  are generally correlated, i.e.,  $P_{X_1, X_2}(x_1, x_2) = f_{\sigma_s}(\mathcal{A}(x_1, x_2))/f_{\sigma_s}(\Lambda)$  (cf. Figure 1). By removing from  $\mathcal{I}'_2$  the bits which are almost deterministic given  $U_2^{1:i-1}$  and  $X_1^{1:N}$ , we obtain the information set  $\mathcal{I}_2$  for  $W_2$ . Then the distribution of the input  $X_2$  becomes the conditional distribution  $P_{X_2|X_1}(x_2|x_1)$ . The process is shown in Figure 5. More precisely, the indices are divided into three portions as follows:

$$\begin{aligned}
 1 &= \underbrace{1 - I(\tilde{X}_2; \tilde{X}_2 \oplus X_2, X_1, Y)}_{|\mathcal{F}_2|/N} \\
 &\quad + I(\tilde{X}_2; \tilde{X}_2 \oplus X_2, X_1, Y) \\
 &\stackrel{\text{Step 1}}{=} \underbrace{1 - I(\tilde{X}_2; \tilde{X}_2 \oplus X_2, X_1, Y)}_{|\mathcal{F}_2|/N} \\
 &\quad + \underbrace{I(\tilde{X}_2; \tilde{X}_2 \oplus X_2)}_{|\mathcal{S}'_2|/N} + \underbrace{I(\tilde{X}_2; X_1, Y | \tilde{X}_2 \oplus X_2)}_{|\mathcal{I}'_2|/N} \\
 &\stackrel{\text{Step 2}}{=} \underbrace{1 - I(\tilde{X}_2; \tilde{X}_2 \oplus X_2, X_1, Y)}_{|\mathcal{F}_2|/N} + \underbrace{1 - H(X_2)}_{|\mathcal{S}'_2|/N} \\
 &\quad + \underbrace{I(X_2; X_1)}_{|\mathcal{S}_{X_1}|/N} + \underbrace{I(X_2; Y | X_1)}_{|\mathcal{I}_2|/N} \\
 &= \underbrace{1 - I(\tilde{X}_2; \tilde{X}_2 \oplus X_2, X_1, Y)}_{|\mathcal{F}_2|/N} + \underbrace{1 - H(X_2 | X_1)}_{|\mathcal{S}_2|/N} \\
 &\quad + \underbrace{I(X_2; Y | X_1)}_{|\mathcal{I}_2|/N}
 \end{aligned}$$

We give the formal statement of this procedure in the following lemma.

**Lemma 9:** After the first step of polarization, we obtain the three sets  $\mathcal{F}_2$ ,  $\mathcal{I}'_2$  and  $\mathcal{S}'_2$  in (19). Let  $\mathcal{S}_{X_1}$  denote the set of indices whose Bhattacharyya parameters satisfy  $Z(U_2^i | U_2^{1:i-1}, Y^{1:N}, X_1^{1:N}) \leq 2^{-N^\beta}$ ,  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}) \leq 1 - 2^{-N^\beta}$  and  $Z(U_2^i | U_2^{1:i-1}) \geq 1 - 2^{-N^\beta}$ . The proportion of  $\mathcal{S}_{X_1}$  is asymptotically given by  $\lim_{N \rightarrow \infty} \frac{|\mathcal{S}_{X_1}|}{N} = I(X_2; X_1)$ . Then by removing  $\mathcal{S}_{X_1}$  from  $\mathcal{I}'_2$ , we obtain the desired information set  $\mathcal{I}_2$  corresponding to the mutual information  $I(X_2; Y | X_1)$  associated with  $W_2$ . Formally, the three sets are obtained as follows:

$$\left\{ \begin{array}{l} \text{frozen set:} \\ \mathcal{F}_2 = \{i \in [N] : Z(U_2^i | U_2^{1:i-1}, Y^{1:N}, X_1^{1:N}) \geq 1 - 2^{-N^\beta}\} \\ \text{information set:} \\ \mathcal{I}_2 = \{i \in [N] : Z(U_2^i | U_2^{1:i-1}, Y^{1:N}, X_1^{1:N}) \leq 2^{-N^\beta} \\ \text{and } Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}) \geq 1 - 2^{-N^\beta}\} \\ \text{shaping set: } \mathcal{S}_2 = (\mathcal{F}_2 \cup \mathcal{I}_2)^c. \end{array} \right. \quad (20)$$

*Proof:* Firstly, we show the proportion of set  $\mathcal{S}_{X_1}$  goes to  $I(X_1; X_2)$  as  $N \rightarrow \infty$ . Here we define a slightly different set  $\mathcal{S}'_{X_1} = \{i \in [N] : Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}) \leq 2^{-N^\beta} \text{ and } Z(U_2^i | U_2^{1:i-1}) \geq 1 - 2^{-N^\beta}\}$ . Suppose we are constructing an asymmetric polar code for the channel from  $X_1$  to  $X_2$ ; it is not difficult to find that  $\lim_{N \rightarrow \infty} \frac{|\mathcal{S}'_{X_1}|}{N} = I(X_2; X_1)$  by Theorem 4. Furthermore, by Lemma 6, if  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}) \leq 2^{-N^\beta}$ , we can immediately have  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}, Y^{1:N}) \leq 2^{-N^\beta}$ . Therefore, the difference between the definitions of  $\mathcal{S}_{X_1}$  and  $\mathcal{S}'_{X_1}$  only lies on  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N})$ . Denoting by  $\bar{\mathcal{P}}_{X_1}$  the unpolarized set with  $2^{-N^\beta} \leq Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}) \leq 1 - 2^{-N^\beta}$ , we have

$$\lim_{N \rightarrow \infty} \left( \frac{|\mathcal{S}_{X_1}|}{N} - \frac{|\mathcal{S}'_{X_1}|}{N} \right) \leq \lim_{N \rightarrow \infty} \frac{|\bar{\mathcal{P}}_{X_1}|}{N} = 0. \quad (21)$$

As a result,  $\lim_{N \rightarrow \infty} \frac{|\mathcal{S}_{X_1}|}{N} = \lim_{N \rightarrow \infty} \frac{|\mathcal{S}'_{X_1}|}{N} = I(X_2; X_1)$ .

Secondly, we show that  $\mathcal{S}_{X_1} \cup \mathcal{I}_2 = \mathcal{I}'_2$ . According to the definitions of  $\mathcal{S}_{X_1}$  and  $\mathcal{I}_2$ , we note that  $\mathcal{S}_{X_1} \cap \mathcal{I}_2 = \emptyset$ . By Lemma 6, if  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N}) \geq 1 - 2^{-N^\beta}$ , we get  $Z(U_2^i | U_2^{1:i-1}) \geq 1 - 2^{-N^\beta}$  and the difference between the definitions of  $\mathcal{S}_{X_1}$  and  $\mathcal{I}'_2$  only lies on  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N})$ . Observe that the union  $\mathcal{S}_{X_1} \cup \mathcal{I}_2$  would remove the condition on  $Z(U_2^i | U_2^{1:i-1}, X_1^{1:N})$ , and accordingly we have  $\mathcal{S}_{X_1} \cup \mathcal{I}_2 = \mathcal{I}'_2$ . It can be also found that the proportion of  $\mathcal{I}_2$  goes to  $I(X_2; Y | X_1)$  as  $N \rightarrow \infty$ .  $\square$

We summarize our main results in the following theorem:

**Theorem 5: (Coding Theorem for Multilevel Polar Codes):** Consider a polar code with the following encoding and decoding strategies for the channel of the second level  $W_2$  with the channel transition probability  $P_{Y|X_2, X_1}(y|x_2, x_1)$  shown in (13).

- **Encoding:** Before sending the codeword  $x_2^{1:N} = u_2^{1:N} G_N$ , the index set  $[N]$  are divided into three parts: the frozen set  $\mathcal{F}_2$ , information set  $\mathcal{I}_2$ , and shaping set  $\mathcal{S}_2$  according to (20). The encoder first places uniformly

distributed information bits in  $\mathcal{I}_2$ . Then the frozen set  $\mathcal{F}_2$  is filled with a uniform random sequence which are shared between the encoder and the decoder. The bits in  $\mathcal{S}_2$  are generated by a mapping  $\phi_{\mathcal{S}_2} \triangleq \{\phi_i\}_{i \in \mathcal{S}_2}$  form a family of randomized mappings  $\Phi_{\mathcal{S}_2}$ , which yields the following distribution:

$$u_2^i = \begin{cases} 0 & \text{with probability } P_{U_2^i|U_2^{1:i-1}, X_1^{1:N}} \\ & \times (0|u_2^{1:i-1}, x_1^{1:N}), \\ 1 & \text{with probability } P_{U_2^i|U_2^{1:i-1}, X_1^{1:N}} \\ & \times (1|u_2^{1:i-1}, x_1^{1:N}). \end{cases} \quad (22)$$

- Decoding: The decoder receives  $y^{1:N}$  and estimates  $\hat{u}_2^{1:N}$  based on the previously recovered  $x_1^{1:N}$  according to the rule

$$\hat{u}_2^i = \begin{cases} u_2^i, & \text{if } i \in \mathcal{F}_2 \\ \phi_i(\hat{u}_2^{1:i-1}), & \text{if } i \in \mathcal{S}_2 \\ \underset{u}{\operatorname{argmax}} P_{U_2^i|U_2^{1:i-1}, X_1^{1:N}, Y^{1:N}} \\ \times (u|\hat{u}_2^{1:i-1}, x_1^{1:N}, y^{1:N}), & \text{if } i \in \mathcal{I}_2. \end{cases}$$

With the above encoding and decoding, the message rate can be arbitrarily close to  $I(Y; X_2|X_1)$  and the expectation of the decoding error probability over the randomized mappings satisfies  $E_{\Phi_{\mathcal{S}_2}}[P_e^{SC}(\phi_{\mathcal{S}_2})] = O(2^{-N^{\beta'}})$  for any  $0 < \beta' < \beta < 0.5$ , where  $\beta$  is used to choose the frozen set, the information set, and the shaping set as in (20).

Note that probability  $P_{U_2^i|U_2^{1:i-1}, X_1^{1:N}, Y^{1:N}}$  can be calculated by (18) efficiently, treating  $\hat{Y}$  and  $X_1$  (already decoded by the SC decoder at level 1) as the outputs of the asymmetric channel. Again, there exists a deterministic mapping  $\phi_{\mathcal{S}_2}$  such that  $P_e^{SC}(\phi_{\mathcal{S}_2}) = O(2^{-N^{\beta'}})$ .

Obviously, Theorem 5 can be generalized to the construction of a polar code for the channel of the  $\ell$ -th level  $W_\ell$ . The only difference is that the side information changes from  $X_1^{1:N}$  to  $X_{1:\ell-1}^{1:N}$ . As a result, we can construct a polar code which achieves a rate arbitrarily close to  $I(Y; X_\ell|X_{1:\ell-1})$  with vanishing error probability.

#### D. Achieving Channel Capacity

So far, we have constructed polar codes to achieve the capacity of the induced asymmetric channels for all levels. Since the sum capacity of the component channels nearly equals the mutual information  $I(Y; X)$ , and since we choose a good constellation such that  $I(Y; X) \approx \frac{1}{2} \log(1 + \text{SNR})$ , we have constructed a lattice code to achieve the capacity of the Gaussian channel. We summarize the construction in the following theorem:

**Theorem 6:** Choose a good constellation with negligible flatness factor  $\epsilon_\Lambda(\tilde{\sigma})$  as in [6, Th. 2], and construct a multilevel polar code with  $r = O(\log \log N)$  as above. Then, for any SNR, the message rate approaches  $\frac{1}{2} \log(1 + \text{SNR})$ , while the error probability under multistage SC decoding is bounded by

$$P_e^{SC} = O(2^{-N^{\beta'}}), \quad 0 < \beta' < 0.5 \quad (23)$$

as  $N \rightarrow \infty$ .

**Remark 4:** It is simple to generate a transmitted codeword of the proposed scheme. For  $n = 1$ , let

$$\chi = \sum_{\ell=1}^r 2^{\ell-1} \left[ \sum_{i \in \mathcal{I}_\ell} u_\ell^i \mathbf{g}_i + \sum_{i \in \mathcal{S}_\ell} u_\ell^i \mathbf{g}_i + \sum_{i \in \mathcal{F}_\ell} u_\ell^i \mathbf{g}_i \right]. \quad (24)$$

The transmitted codeword  $x$  is drawn from  $D_{2^r \mathbb{Z}^N + \chi, \sigma_s}$ . From the proof of Lemma 5, we know that the probability of choosing a point outside of the interval  $[-2^{r-1}, 2^{r-1}]$  is negligible if  $r$  is sufficiently large, which implies there exists only one point in this interval with probability close to 1. Therefore, one may simply transmit  $x = \chi \bmod 2^r$ , where the modulo operation is applied component-wise with range  $(-2^{r-1}, 2^{r-1}]$ .

Next, we show that such a multilevel polar coding scheme is equivalent to Gaussian shaping over a coset  $L + c'$  of a polar lattice  $L$  for some translate  $c'$ . In fact, the polar lattice  $L$  is exactly constructed from the corresponding symmetrized channels  $\tilde{W}_\ell$ . Recall that the  $\ell$ -th channel  $W_\ell$  is a BMA channel with the input distribution  $P_{X_\ell|X_{1:\ell-1}}$  ( $1 \leq \ell \leq r$ ). It is clear that  $P_{X_{1:\ell}}(x_{1:\ell}) = f_{\sigma_s}(\mathcal{A}_\ell(x_{1:\ell})) / f_{\sigma_s}(\Lambda)$ . By Lemma 7 and (13), the transition probability of the symmetrized channel  $\tilde{W}_\ell$  is

$$\begin{aligned} & P_{\tilde{W}_\ell}((y, x_{1:\ell-1}, x_\ell \oplus \tilde{x}_\ell) | \tilde{x}_\ell) \\ &= P_{Y, X_{1:\ell}}(y, x_{1:\ell}) \\ &= P_{X_{1:\ell}}(x_{1:\ell}) P_{Y|X_\ell, X_{1:\ell-1}}(y | x_\ell, x_{1:\ell-1}) \\ &= \exp\left(-\frac{\|y\|^2}{2(\sigma_s^2 + \sigma^2)}\right) \frac{1}{f_{\sigma_s}(\Lambda)} \frac{1}{2\pi\sigma\sigma_s} \sum_{a \in \mathcal{A}_\ell(x_{1:\ell})} \\ & \times \exp\left(-\frac{\|\alpha y - a\|^2}{2\tilde{\sigma}^2}\right). \end{aligned} \quad (25)$$

Note that the difference between the asymmetric channel (13) and symmetrized channel (25) is the *a priori* probability  $P_{X_{1:\ell}}(x_{1:\ell}) = f_{\sigma_s}(\mathcal{A}_\ell(x_{1:\ell})) / f_{\sigma_s}(\Lambda)$ . Comparing with the  $\Lambda_{\ell-1}/\Lambda_\ell$  channel (10), we see that the symmetrized channel (25) is equivalent to a  $\Lambda_{\ell-1}/\Lambda_\ell$  channel, since the common terms in front of the sum will be completely cancelled out in the calculation of the likelihood ratio.<sup>2</sup> We summarize the foregoing analysis in the following lemma:

**Lemma 10: (Equivalence Lemma):** Consider a multilevel lattice code constructed from constellation  $D_{\Lambda, \sigma_s}$  for a Gaussian channel with noise variance  $\sigma^2$ . The  $\ell$ -th symmetric channel  $\tilde{W}_\ell$  ( $1 \leq \ell \leq r$ ) which is derived from the asymmetric channel  $W_\ell$  is equivalent to the MMSE-scaled  $\Lambda_{\ell-1}/\Lambda_\ell$  channel with noise variance  $\tilde{\sigma}^2$ .

Thus, the resultant polar codes for the symmetrized channels are nested, and the polar lattice is AWGN-good for noise variance  $\tilde{\sigma}^2$ ; also, the multistage decoding is performed on the MMSE-scaled signal  $\alpha y$  (cf. Lemma 8). Since the frozen sets of the polar codes are filled with random bits (rather than all zeros), we actually obtain a coset  $L + c'$  of the polar lattice, where the shift  $c'$  accounts for the effects of all random frozen bits. Finally, since we start from  $D_{\Lambda, \sigma_s}$ , we would obtain  $D_{\Lambda^N, \sigma_s}$  without coding; since  $L + c' \subset \Lambda^N$

<sup>2</sup>Even if  $y \in \mathbb{R}^n$  in (25), the sum over  $\mathcal{A}_\ell(x_{1:\ell})$  is  $\Lambda_\ell$ -periodic. Hence, the likelihood ratio will be the same if one takes  $\tilde{y} = y \bmod \Lambda_\ell$  and uses (10)

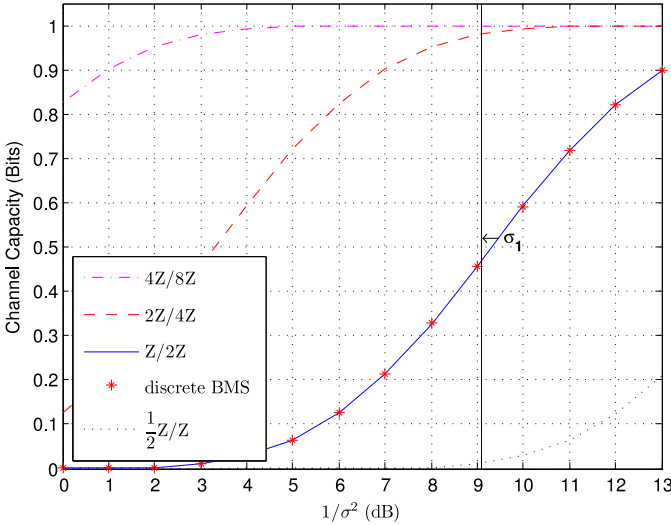


Fig. 6. Channel capacity for partition chain  $\mathbb{Z}/2\mathbb{Z}/\dots/2^r\mathbb{Z}$ . The discrete BMS approximation uses the quantization-merging algorithm with 64 quantization levels.

by construction, we obtain a discrete Gaussian distribution  $D_{L+c',\sigma_s}$  over  $L+c'$ .

*Remark 5:* This analysis shows that our proposed scheme is an explicit construction of lattice Gaussian coding introduced in [6], which applies Gaussian shaping to an AWGN-good lattice (or its coset). Note that the condition of negligible  $\epsilon_\Lambda(\tilde{\sigma})$  in Theorem 6 implies negligible capacity  $C(\Lambda,\tilde{\sigma}^2)$  of the top lattice in the construction of the AWGN-good lattice in Section III. Again, it is always possible to scale down the top lattice  $\Lambda$  such that  $\epsilon_\Lambda(\tilde{\sigma})$  becomes negligible. Thus, Theorem 6 holds for any SNR, meaning that we have removed the condition  $\text{SNR} > e$  required by [6, Th. 3]. Moreover, if a good constellation of the form  $D_{\Lambda-c,\sigma_s}$  for some shift  $c$  is used in practice (e.g., a constellation taking values in  $\{\pm 1, \pm 3, \dots\}$ ), the proposed construction holds verbatim.

## V. DESIGN EXAMPLES

In this section, we give design examples of polar lattices with and without the power constraint. The design follows the equal-error-probability rule. Multistage SC decoding is applied. Since the complexity of SC decoding is  $O(N \log N)$ , the overall decoding complexity is  $O(rN \log N)$ .

### A. Design Examples Without Power Constraint

Consider the one-dimensional lattice partition  $\mathbb{Z}/2\mathbb{Z}/\dots/2^r\mathbb{Z}$ . To construct a multilevel lattice, one needs to determine the number of levels of lattice partitions and the actual rates according to the target error probability for a given noise variance. By the guidelines given in Section III, the effective levels are those which can achieve the target error probability with an actual rate not too close to either 0 or 1. Therefore, one can determine the number of effective levels with the help of capacity curves in Fig. 6. For example, for the given noise variance indicated by the straight line in Fig. 6, one may choose partition  $\mathbb{Z}/2\mathbb{Z}/4\mathbb{Z}$ , i.e.,  $r = 2$ , which was indeed suggested in [17].

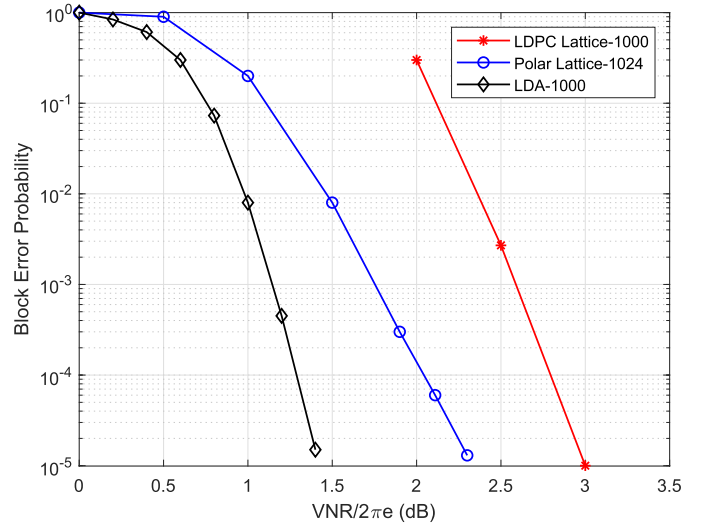


Fig. 7. Performance comparison of lattices at dimension around 1000.

Now, we give an example for length  $N = 1024$  and target error probability  $P_e(L, \sigma^2) = 10^{-5}$ . We note that the calculation of  $C(\Lambda/\Lambda', \sigma^2)$  and  $P_e(\Lambda, \sigma^2)$  can be simplified by the scaling property of the partition channels as shown in the proof of Lemma 3. For the one-dimensional partition chain, we have  $C(4\mathbb{Z}/8\mathbb{Z}, \sigma^2) = C(2\mathbb{Z}/4\mathbb{Z}, (\frac{\sigma}{2})^2) = C(\mathbb{Z}/2\mathbb{Z}, (\frac{\sigma}{4})^2)$ , and  $P_e(4\mathbb{Z}, \sigma^2) = P_e(2\mathbb{Z}, (\frac{\sigma}{2})^2) = P_e(\mathbb{Z}, (\frac{\sigma}{4})^2)$ . Let  $\sigma_1 = \sigma$ ,  $\sigma_2 = \sigma/2$  and  $\sigma_3 = \sigma/4$  be the equivalent Gaussian noise deviation at the  $\ell$ 's level with respect to the 1st one.

Since the bottom level is a  $\mathbb{Z}^N$  lattice decoder,  $\sigma_3 \approx 0.0845$  for target error probability  $\frac{1}{3} \cdot 10^{-5}$ . For the middle level,  $\sigma_2 = 2 \cdot \sigma_3 = 0.1690$ . From Fig. 6, the channel capacity of the middle level is  $C(\mathbb{Z}/2\mathbb{Z}, \sigma_2^2) = C(2\mathbb{Z}/4\mathbb{Z}, \sigma_1^2) = 0.9874$ . For the top level,  $\sigma = \sigma_1 = 0.3380$  and the capacity is 0.5145. Our goal is to find two polar codes approaching the respective capacities at block error probabilities  $\leq \frac{1}{3} \cdot 10^{-5}$  over these binary-input mod-2 channels.

For  $N = 1024$ , we found the first polar code with rate  $\frac{k_1}{N} = 0.23$  for  $P_e(\mathcal{C}_1, \sigma_1^2) \approx \frac{1}{3} \cdot 10^{-5}$ , and the second polar code with rate  $\frac{k_2}{N} = 0.9$  for  $P_e(\mathcal{C}_2, \sigma_2^2) \approx \frac{1}{3} \cdot 10^{-5}$ . Recall that the channel in the first level is degraded with respect to the one at the second level according to Lemma 3, and the two polar codes in this construction turn out to be nested. Thus, the sum rate of component polar codes  $R_C = 0.23 + 0.9$ , implying a capacity loss  $\epsilon_3 = 0.3719$ . Meanwhile, the factor  $\epsilon_1 = C(\mathbb{Z}, 0.3380^2) = 0.0160$ . Therefore, the rate losses at each level are 0.016, 0.285, and 0.087. From (8), the logarithmic VNR is given by

$$\log \left( \frac{\gamma_L(\sigma)}{2\pi e} \right) \leq 2(\epsilon_1 + \epsilon_3) = 0.7758, \quad (26)$$

which is 2.34 dB. It is seen from Fig. 7 that the estimate 2.34 dB is very close to the actual gap at  $P_e(L, \sigma_1^2) \approx 10^{-5}$ . This simulation indicates that the gap to the Poltyrev capacity is largely due to the capacity losses of component codes.

Performance comparison of competing lattices approaching the Poltyrev capacity is presented in Fig. 7, at dimension around 1000. The polar lattice used here is constructed from

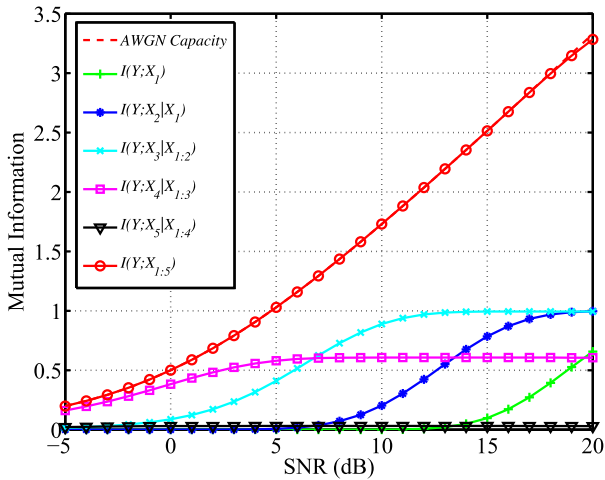


Fig. 8. Channel capacity for each level as a function of SNR.

the aforementioned one-dimensional lattice partition ( $N = 1024, n = 1$ ). The simulation curves of other lattices are taken from their corresponding papers. Among the three types of lattices compared, the LDPC lattice [12] has the weakest performance. The LDA lattice [13] has better performance than the polar lattice, at the expense of higher decoding complexity  $O(p^2 N \log N)$  if  $p$ -ary LDPC codes are employed. Assuming  $p \approx 2^r$ , it would require complexity  $O(2^{2r} N \log N)$ , compared to  $O(rN \log N)$  of the polar lattice. The LDLC lattice is not included in this comparison because of lack of block error probabilities in [14]. In contrast to the polar lattice and LDA lattice, analytic results of the LDLC are not available; therefore, they are less understood in theory. It is worth pointing out that the plain polar codes used in polar lattice can be optimized in several aspects: for example, to use a better kernel, list decoding, or even a soft-output decoding algorithm. We leave such improvements of polar lattices to future work.

### B. Design Examples With Power Constraint

To satisfy the power constraint, we use discrete lattice distribution  $D_{\mathbb{Z}_s, \sigma_s}$  for shaping. The mutual information  $I(Y; X_\ell | X_{1:\ell-1})$  at each level for different SNRs is shown in Figure 8. We can see that for partition  $\mathbb{Z}/2\mathbb{Z}/\dots$ , five levels are enough to achieve the AWGN channel capacity for SNR ranging from  $-5$  dB to 20 dB. Note that the actual number of required levels depends on the SNR: a smaller number of levels are enough for low SNRs, while a larger number of levels is required for high SNRs (to support higher rates).

For each level, we estimate a lower bound on the code rate for block error probability  $1 \times 10^{-5}$ . This is done by calculating an upper bound on the block error probability of the polar code, using the Bhattacharyya parameter. With this target error probability, the assignments of bits to the information, shaping and frozen sets on different levels are shown in Figure 9 for SNR = 15 dB and  $N = 2^{16}$ . In fact,  $X_1$  and  $X_2$  are nearly uniform such that there is no need for shaping on the first two levels (these levels actually correspond to the AWGN-good lattice). The third level channel is very

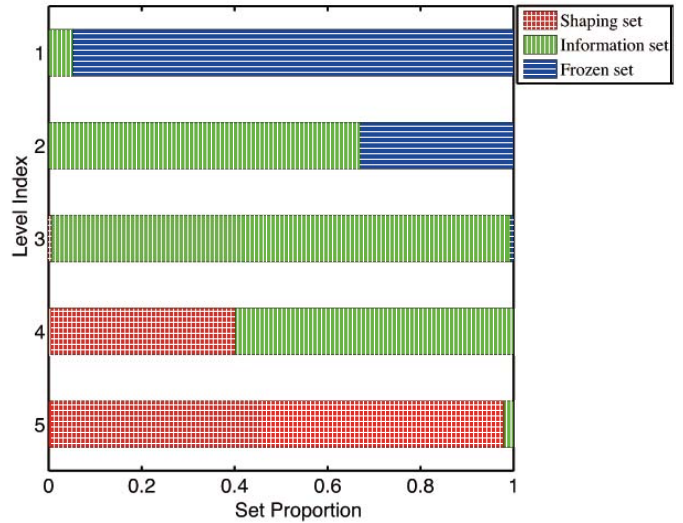


Fig. 9. The proportions of the shaping set, information set, and frozen set on each level when  $N = 2^{16}$  and SNR = 15 dB.

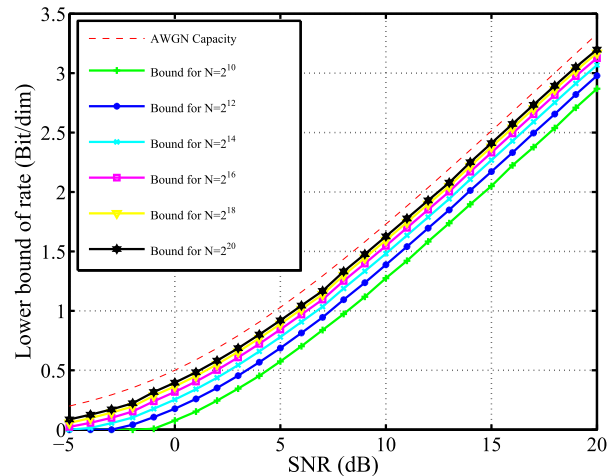


Fig. 10. Lower bounds on the rates achieved by polar lattices with block error probability  $5 \times 10^{-5}$  for block lengths  $2^{10}, \dots, 2^{20}$ .

clean, and most bits are information bits. In contrast, the fifth level is mostly for shaping; since its message rate is already small, adding another level clearly would not contribute to the overall rate of the lattice code. Finally, lower bounds on the rates achieved by polar lattices with various block lengths are shown in Figure 10. We note that the gap to the channel capacity diminishes as  $N$  increases, and it is only about 0.1 bits/dimension when  $N = 2^{20}$ .

## VI. CONCLUSIONS

In this paper, we have constructed polar lattices to approach the capacity of the power-constrained Gaussian channel. The construction is based on a combination of channel polarization and source polarization. Without shaping, the constructed polar lattices are AWGN-good. The Gaussian shaping on a polar lattice deals with the power constraint but is technically more involved. Our shaping approach is different from the standard Voronoi shaping which involves a quantization-good

lattice [5]. The proposed Gaussian shaping does not require such a quantization-good lattice any more. The overall scheme is explicit and efficient, featuring quasi-linear complexity.

#### APPENDIX A PROOF OF LEMMA 2

*Proof:* For this purpose, we assume  $\Lambda = a\mathbb{Z}^n$  and  $\Lambda' = b\mathbb{Z}^n$  where  $a, b$  are scaling parameters to be estimated. We note that for all partition chains in [17], this is always possible: if the bottom lattice does not take the form of  $b\mathbb{Z}^n$ , one may simply further extend the partition chain (which will lead to an upper bound on  $r$ ).

We firstly note that the flatness factor  $\epsilon_\Lambda(\sigma)$  can be made arbitrarily small by scaling down the top lattice  $\Lambda$ . To see this, we recall that  $\epsilon_\Lambda(\sigma_e) \leq [1 + \epsilon_{\Lambda_0}(\sigma_e)]^n - 1$  [6, Lemma 3] where  $\Lambda_0 = a\mathbb{Z}$  for the afore-mentioned scaling factor  $a$ .

Let  $\Lambda_0^* = \frac{1}{a}\mathbb{Z}$  be the dual lattice of  $\Lambda_0$ . By [18, Corollary 1], we have

$$\begin{aligned} \epsilon_{\Lambda_0}(\sigma) &= \Theta_{\Lambda_0^*}(2\pi\sigma^2) - 1 \\ &= \sum_{\lambda \in \Lambda_0^*} \exp(-2\pi^2\sigma^2|\lambda|^2) - 1 \\ &= 2 \sum_{\lambda \in \frac{1}{a}\mathbb{N}} \exp(-2\pi^2\sigma^2|\lambda|^2) \\ &\leq \frac{2 \exp(-2\pi^2\sigma^2 \frac{1}{a^2})}{1 - \exp(-2\pi^2\sigma^2 \frac{3}{a^2})} \\ &\leq 4 \exp(-2\pi^2\sigma^2 \frac{1}{a^2}) \text{ for sufficiently small } a. \end{aligned} \quad (27)$$

Therefore, letting  $\frac{1}{a} = \sqrt{\frac{N}{2\pi^2\sigma^2}}$ , we have  $\epsilon_{\Lambda_0}(\sigma) = O(e^{-N})$  and hence  $\epsilon_\Lambda(\sigma) = O(e^{-N})$  for fixed  $n$ .

Secondly, by the union bound, the error probability of the bottom lattice  $\Lambda'$  is upper-bounded by

$$P_e(\Lambda', \sigma^2) \leq nQ\left(\frac{b}{2\sigma}\right) \leq ne^{-\frac{b^2}{8\sigma^2}}$$

where we apply the Chernoff bound on the Q-function. We can obtain

$$P_e(\Lambda', \sigma^2) = O(e^{-N})$$

by choosing  $b = \sqrt{8\sigma^2 N}$  for fixed  $n$ .

For a binary lattice partition, we have  $(b/a)^n = 2^r$ . Thus, we conclude that

$$r = n \log\left(\frac{b}{a}\right) = n \log\left(\frac{2}{\pi}\right) \leq n \log N = O(\log N). \quad \square$$

#### APPENDIX B PROOF OF LEMMA 3

*Proof:* By the self-similarity of the lattice partition chain, we can scale a  $\Lambda_{\ell-1}/\Lambda_\ell$  channel to a  $\Lambda_\ell/\Lambda_{\ell+1}$  channel by multiplying the output of a  $\Lambda_{\ell-1}/\Lambda_\ell$  channel with  $T$ . Since  $T = \alpha V$  for some scaling factor  $\alpha > 1$  and orthogonal matrix  $V$ , the Gaussian noise for each dimension is still independent of each other and the noise variance per dimension is increased after the scaling. Therefore, a  $\Lambda_{\ell-1}/\Lambda_\ell$  channel is

stochastically equivalent to a  $\Lambda_\ell/\Lambda_{\ell+1}$  channel with a larger noise variance. For our design examples, a  $\mathbb{Z}/2\mathbb{Z}$  channel with Gaussian noise variance  $\sigma^2$  is equivalent to a  $2\mathbb{Z}/4\mathbb{Z}$  channel with Gaussian noise variance  $4\sigma^2$ , and a  $\mathbb{Z}^2/R\mathbb{Z}^2$  channel with noise variance  $\sigma^2$  per dimension is equivalent to a  $R\mathbb{Z}^2/2\mathbb{Z}^2$  channel with noise variance  $2\sigma^2$  per dimension. Proof is completed by observing that a  $\Lambda_\ell/\Lambda_{\ell+1}$  channel with noise variance  $\sigma_2^2$  is degraded with respect to a  $\Lambda_\ell/\Lambda_{\ell+1}$  channel with noise variance  $\sigma_1^2$  if  $\sigma_1^2 \leq \sigma_2^2$ .  $\square$

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